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# Cardinal Interpolation and Spline Functions: II
# Interpolation of Data of Power Growth*†

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Let $m$ be a natural number and let $S_m$ denote the class of functions $S(x)$ of the following nature: If $m$ is even, then $S(x)$ is a polynomial of degree $m - 1$ in each unit interval $(\nu, \nu + 1)$ for all integer values of $\nu$, while $S(x) \in C^{m-2}$ on the entire real axis. If $m$ is odd, then the conditions are the same except that the intervals $(\nu, \nu + 1)$ are replaced by $(\nu - \frac{1}{2}, \nu + \frac{1}{2})$. The main result is as follows: If a sequence $(y_\nu)(-\infty < \nu < \infty)$ of numbers is preassigned such that $y_\nu = O(|\nu|^s)$ as $\nu \to \pm\infty$, with $s \geqslant 0$, then there exists a unique $S(x) \in S_m$ satisfying the relations $S(\nu) = y_\nu$, for all integer $\nu$, and the growth condition $S(x) = O(|x|^s)$ as $x \to \pm\infty$.

## INTRODUCTION AND STATEMENT OF RESULTS

Let $m$ be a natural number and let $S_m$ denote the class of spline functions of degree $m - 1$, or order $m$, defined on the entire axis of reals and having simple knots at the integers $\nu$ if $m$ is even, or at the half-way points $\nu + \frac{1}{2}$ if $m$ is odd. This means that $S(x) \in S_m$ provided that $S(x) \in C^{m-2}$ (for $m = 1$ this condition is vacuous) and that the restriction of $S(x)$ to any interval between consecutive knots is identical with a polynomial of degree not exceeding $m - 1$. Such functions are called *cardinal spline functions.*

Let $y = (y_\nu)$ be a prescribed sequence of numbers, the subscript $\nu$ ranging over all integers. The problem of finding a function $F(x)$ $(-\infty < x < \infty)$ satisfying the relations

$$F(\nu) = y_\nu \qquad \text{for all} \quad \nu, \tag{1}$$

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and such that $F(x)$ belongs to some prescribed linear space $\mathscr{S}$ is called a *cardinal interpolation problem* and denoted by the symbol

$$\mathrm{CIP}(y; \mathscr{S}). \tag{2}$$

Before we proceed with the contents of this paper it is important for both motivation and orientation to state and prove the following

LEMMA 1. *Let $m \geqslant 2$ and let the sequence $(y_\nu)$ be arbitrarily assigned. The*

$$\mathrm{CLP}\,(y; S_m) \tag{3}$$

*always has solutions and all its solutions form a linear manifold of dimension* $2[(m-1)/2]$, *i.e., its dimension is $m-2$ if $m$ is even and $m-1$ if $m$ is odd.*

*Proof.* Let us construct a solution $S(x)$ of the problem (3).

*Case* 1. *$m$ is even.* The knots of $S(x)$ are at the integers. Choose $S(x) = P(x) \in \pi_{m-1}$ in $0 \leqslant x \leqslant 1$, the polynomial $P(x)$ being only required to satisfy the conditions

$$P(0) = y_0 \quad \text{and} \quad P(1) = y_1 . \tag{4}$$

The entire solution $S(x)$ may now be written in the form

$$\begin{aligned} S(x) = {} & P(x) + a_1(x-1)_+^{m-1} + a_2(x-2)_+^{m-1} \\ & + \cdots + a_0(-x)_+^{m-1} + a_{-1}(-x-1)_+^{m-1} + \cdots , \end{aligned} \tag{5}$$

where $u_+ = u$ if $u \geqslant 0$ and $u_+ = 0$ if $u < 0$. Observe that the interpolation conditions

$$S(\nu) = y_\nu \quad \text{for} \quad \nu = 2, 3, \ldots$$

determine successively and uniquely the coefficients $a_1, a_2, \ldots$, respectively, while the conditions

$$S(\nu) = y_\nu \quad \text{for} \quad \nu = -1, -2, \ldots,$$

likewise determine successively and uniquely the coefficients $a_0, a_{-1}, \ldots$, respectively. Thus, indeed, the solution $S(x)$ is uniquely defined by the choice of the polynomial $P(x)$. Since $P(x)$ depends on $m-2$ linear parameters, the proof for this case is complete.

*Case* 2. *$m$ is odd.* The knots are now at the points $\nu + \frac{1}{2}$. We choose $S(x) = Q(x) \in \pi_{m-1}$, in $-\frac{1}{2} \leqslant x \leqslant \frac{1}{2}$, requiring $Q(x)$ to satisfy

$$Q(0) = y_0 . \tag{6}$$

After this choice the solution of (3) is uniquely defined by

$$S(x) = Q(x) + b_1(x - \tfrac{1}{2})_+^{m-1} + b_2(x - \tfrac{3}{2})_+^{m-1}$$
$$+ \cdots + b_0(-x - \tfrac{1}{2})_+^{m-1} + b_{-1}(-x - \tfrac{3}{2})_+^{m-1} + \cdots,$$

where the coefficients are uniquely and successively determined from $S(\nu) = y_\nu$ ($\nu \neq 0$) as before. As $Q(x)$, subject to (6), depends on $m - 1$ linear parameters, the proof of Lemma 1 is complete.

If $m = 2$, which is the case of linear spline functions, the problem (3) has a unique solution.

If $m \geqslant 3$, then most of the solutions of (3) are perfectly wild and useless functions. It is clear that some further useful restrictions on $(y_\nu)$ and on the interpolant $S(x)$ are needed. Two such restrictions were given in [4] by prescribing the following two choices for the space $\mathscr{S}$:

$$\mathscr{S}_1 = S_{m+1} \cap L_p^m \qquad \text{and} \qquad \mathscr{S}_2 = S_{2m} \cap L_2^m. \tag{7}$$

Here $1 \leqslant p \leqslant \infty$ and $L_p^m$ is the class of functions $F(x)$ such that $F^{(m)}(x) \in L_p$. It was shown that both these problems have solutions, and then unique solutions, if and only if the sequence $(y_\nu)$ satisfies the condition

$$\Delta^m y = (\Delta^m y_\nu) \in l_p. \tag{8}$$

The present paper is a supplement to [4]. We discuss first the problem (2) for more general and perhaps also more useful spaces $\mathscr{S}$, namely, the following: Let $s \geqslant 0$ and let us consider the class

$$F_s = \{F(x); F(x) \in C, \qquad F(x) = O(|x|^s) \qquad \text{as} \quad x \to \pm\infty\}. \tag{9}$$

In particular, $F_0$ is the class of bounded and continuous functions. We may also describe the class

$$F^* = \bigcup_{s \geqslant 0} F_s \tag{10}$$

as the class of functions of power growth.

Correspondingly, let

$$Y_s = \{y = (y_\nu); \qquad y_\nu = O(|\nu|^s) \qquad \text{as} \quad \nu \to \pm\infty\} \tag{11}$$

and

$$Y^* = \bigcup_{s \geqslant 0} Y_s. \tag{12}$$

Our present main results on cardinal spline interpolation are the following theorems.

THEOREM 1. *The*

$$\mathrm{CIP}(y; S_m \cap F_s) \tag{13}$$

*has solutions if and only if*

$$y \in Y_s \tag{14}$$

*and then the problem* (13) *has a unique solution.*

This theorem shows that *the interpolation conditions* (1) *set up a one-to-one correspondence between the elements of the two classes*

$$S_m \cap F^* \quad \text{and} \quad Y^*.$$

Let us now consider the unit-sequence

$$\delta = (\delta_\nu), \qquad \text{where} \quad \delta_\nu = \begin{cases} 1 & \text{if} \quad \nu = 0, \\ 0 & \text{if} \quad \nu \neq 0. \end{cases} \tag{15}$$

Evidently $\delta \in Y_0$ and by Theorem 1 the CIP $(\delta; S_m \cap F_0)$ has a unique solution that we denote by $L_m(x)$. Thus

$$L_m(\nu) = \delta_\nu \qquad \text{for all} \quad \nu. \tag{16}$$

THEOREM 2. *The function $L_m(x)$ satisfies an inequality of the form*

$$| L_m(x)| < C_m \exp(-\gamma_m | x |) \qquad \textit{for all real} \quad x, \tag{17}$$

*where $C_m$ and $\gamma_m$ are positive constants.*

THEOREM 3. *If the condition* (14) *is satisfied, then the unique solution $S(x)$ of the problem* (13) *is given by "Lagrange's formula"*

$$S(x) = \sum_{-\infty}^{\infty} y_\nu L_m(x - \nu) \tag{18}$$

*which converges locally uniformly.*

We note the

COROLLARY 1. If *$S(x) \in S_m \cap F^*$ then the relation*

$$S(x) = \sum_{-\infty}^{\infty} S(\nu)\, L_m(x - \nu) \tag{19}$$

*is an identity.*

Because of its role in Theorem 3 we call $L_m(x)$ the *fundamental cardinal spline function* of order $m$, or degree $m - 1$.

All these theorems are evident if $m = 1$, or $m = 2$. In particular, $L_2(x)$ is the well-known "roof-function"

$$L_2(x) = \begin{cases} 1 + x & \text{in } [-1, 0] \\ 1 - x & \text{in } [0, 1] \\ 0 & \text{elsewhere} \end{cases}$$

and the formula (18) gives the unique interpolant in $S_2$ for perfectly arbitrary data $(y_\nu)$. The theorems are no longer evident if $m \geqslant 3$. For instance, it is by no means trivial that if $(y_\nu)$ is a bounded sequence then there is a unique interpolating function $S(x)$ satisfying

$$S(x) \in C^1, \qquad S(x) \in F^*,$$

and that reduces to a quadratic function on each interval $(\nu - \frac{1}{2}, \nu + \frac{1}{2})$ for all integers $\nu$. Moreover, this unique $S(x)$ is bounded.

For the origin of the term "cardinal" in the present context, see [3, pp. 46–47].

Before we mention the second subject of this paper, we recall the following known facts. Let

$$M_1(x) = \begin{cases} 1 & \text{if } -\frac{1}{2} \leqslant x \leqslant \frac{1}{2} \\ 0 & \text{elsewhere} \end{cases} \tag{20}$$

and let

$$M_m(x) = \overbrace{M_1 * \cdots * M_1}^{m}(x) \tag{21}$$

be the result of convoluting $M_1(x)$ with itself $m$ times. It is easily seen that

$$M_m(x) \in S_m \tag{22}$$

and that the support of $M_m(x)$ is contained in $(-\frac{1}{2}m, \frac{1}{2}m)$. The spline function $M_m(x)$ is called a basis-spline, or $B$-spline, because of the following property [3, Part A, Section 3.15, Theorem 5]. If

$$S(x) \in S_m\,, \tag{23}$$

then $S(x)$ admits a unique representation of the form

$$S(x) = \sum_{-\infty}^{\infty} c_\nu M_m(x - \nu) \tag{24}$$

and, conversely, any such series with arbitrarily prescribed $(c_\nu)$ converges and (23) holds.

Notice that $M_m(x) = L_m(x)$ if $m = 1$ or $m = 2$, but this is no longer true if $m \geqslant 3$.

Let (23) and (24) hold. When is $S(x) \in L_p$ ? An answer is provided by Theorem 12 of [4] which we restate here as

THEOREM 4. *Let* (23) *and* (24) *hold and let* $1 \leqslant p \leqslant \infty$. *Then*

$$S(x) \in L_p \tag{25}$$

*if and only if*

$$(c_\nu) \in l_p\,. \tag{26}$$

We give here a new proof of Theorem 4. It will rather easily follow from the following result that is perhaps of independent interest.

THEOREM 5. *Let* $(\gamma_{2r}^{(m)})$ *be the sequence of rational numbers generated by the expansion*

$$\left(\frac{u}{2\sin(u/2)}\right)^m = \sum_{r=0}^{\infty} \gamma_{2r}^{(m)} u^{2r}. \tag{27}$$

*If* (23) *and* (24) *hold, then the coefficients* $c_\nu$ *in* (24) *may be expressed as*

$$c_\nu = \sum_{r=0}^{s} (-1)^r \gamma_{2r}^{(m)} S^{(2r)}(\nu), \qquad \text{where} \quad s = [(m-1)/2]. \tag{28}$$

This may be regarded as an analogue for cardinal spline functions of Taylor's formula for polynomials.

From (27) we find that

$$\gamma_2^{(m)} = m/24, \qquad \gamma_4^{(m)} = m(5m+2)/5760,$$

whence (28) assumes the following explicit forms:

$$\begin{aligned}
m &= 2, \qquad c_\nu = S(\nu),\\
m &= 3, \qquad c_\nu = S(\nu) - (1/8)\, S''(\nu),\\
m &= 4, \qquad c_\nu = S(\nu) - (1/6)\, S''(\nu),\\
m &= 5, \qquad c_\nu = S(\nu) - (5/24)\, S''(\nu) + (3/128)\, S^{(4)}(\nu),\\
m &= 6, \qquad c_\nu = S(\nu) - \tfrac{1}{4} S''(\nu) + (1/30)\, S^{(4)}(\nu).
\end{aligned}$$

In Section 1 below we describe a few applications of Theorems 1 and 3. A characterization of periodic spline interpolants to periodic data follows very easily (Theorem 6). The most beautiful and most important spline functions and monosplines are the so-called *Euler splines* $\mathcal{E}_m(x)$ and the *Bernoulli monosplines* $\bar{B}_m(x)$. New characterizations of these functions are obtained (Theorems 7 and 8).

In Sections 2–7, Theorems 1–5 are established by using some results from [3, 4].

## 1. A Few Applications

### A. *The case of periodic sequences* $(y_\nu)$

Let us apply Theorem 1 to the case when

$$(y_\nu) \text{ is a periodic sequence of period } n. \tag{1.1}$$

Since $(y_\nu) \in Y_0$ it follows that the CIP $(y; S_m \cap F_0)$ has a unique solution. Since $S(\nu) = y_\nu$ for all $\nu$, (1.1) implies that $S(\nu + n) = y_{\nu+n} = y_\nu$, or

$$S(\nu + n) = y_\nu$$

for all $\nu$. Therefore $S(x + n)$ is another solution of our problem. From the unicity we conclude that $S(x + n) = S(x)$. This establishes

THEOREM 6. *If $(y_\nu)$ is a periodic sequence of period $n$, then the unique solution of the*

$$\mathrm{CIP}(y; S_m \cap F^*) \tag{1.2}$$

*is periodic of period $n$.*

The periodicity of $S(x)$ becomes immediately apparent if we use its Lagrange representation (18). In our case, (18) becomes

$$S(x) = \sum_{0}^{n-1} y_\nu l_\nu(x), \tag{1.3}$$

where

$$l_\nu(x) = \sum_{j=-\infty}^{\infty} L_m(x - \nu - jn) \tag{1.4}$$

are $n$ fixed periodic elements of $S_m$ that form a base for all such functions.

B. *The Euler splines*

We consider the simplest nontrivial periodic sequence

$$y_\nu = (-1)^\nu \qquad \text{for all} \quad \nu. \tag{1.5}$$

By Theorem 6 we obtain

THEOREM 7. *The*

$$\mathrm{CLP}\,((-1)^\nu\,;\, S_{m+1} \cap F^*) \tag{1.6}$$

*has a unique solution denoted by* $\mathscr{E}_m(x)$ *and called the Euler spline of degree* $m$; $\mathscr{E}_m(x)$ *is periodic of period* 2.

This is a new characterization of the Euler spline $\mathscr{E}_m(x)$ within the class $S_{m+1}$. An entirely different characterization of $\mathscr{E}_m(x)$ was recently discussed by Cavaretta [1].

We have already mentioned the relations

$$\mathscr{E}_m(\nu) = (-1)^\nu \qquad \text{for all} \quad \nu, \tag{1.7}$$

and

$$\mathscr{E}_m(x + 2) = \mathscr{E}_m(x). \tag{1.8}$$

From (1.7) we get $-\mathscr{E}_m(\nu - 1) = (-1)^\nu$ whence the unicity shows that

$$\mathscr{E}_m(x - 1) = -\mathscr{E}_m(x). \tag{1.9}$$

We could develop the further properties of $\mathscr{E}_m(x)$ from its present definition. However, for brevity we only remark the following: Nörlund [2, Section 2] discusses the properties of the Euler polynomials $E_m(x)$, defined as the solutions of the identity $E_m(x + 1) + E_m(x) = 2x^m$, and forms the periodic function $\bar{E}_m(x)$ of period 2. From Nörlund's discussion it is seen that also $\bar{E}_m(x)$ is a spline function of degree $m$ having its knots at the integers. From the unicity of the solution of the problem (1.6) we conclude that

$$\mathscr{E}_m(x) = \begin{cases} \bar{E}_m(x + \frac{1}{2})/\bar{E}_m(\frac{1}{2}) & \text{if } m \text{ is even} \\ \bar{E}_m(x)/\bar{E}_m(0) & \text{if } m \text{ is odd.} \end{cases} \tag{1.10}$$

For the role of the Euler splines in Kolmogorov's solution of Landau's problem for the real axis see [6, Section 1].

C. *The Bernoulli monosplines*

We consider the

$$\mathrm{CIP}(\nu^m; S_m \cap F^*) \tag{1.11}$$

and observe that by Theorems 1 and 3 its unique solution is the spline function

$$S(x) = \sum_{-\infty}^{\infty} \nu^m L_m(x - \nu). \tag{1.12}$$

Since $S(x)$ interpolates $x^m$ at the integers, if we define the remainder $R(x)$ by

$$x^m = S(x) + R(x), \tag{1.13}$$

then

$$R(\nu) = 0 \qquad \text{for all} \quad \nu. \tag{1.14}$$

We may restate our result as follows: First we recall that a function of the form

$$R(x) = x^m - S(x), \qquad \text{where} \quad S(x) \in S_m \tag{1.15}$$

is called a *cardinal monospline* of degree $m$. Besides (1.14) our monospline $R(x)$ also satisfies

$$R(x) \in F^*. \tag{1.16}$$

We also know, by Theorem 1, that the monospline $R(x)$ is uniquely characterized, among monosplines, by the properties (1.14) and (1.16).

However, a monospline satisfying these conditions was known for a long time: Let $B_m(x)$ be the Bernoulli polynomial and let $\bar{B}_m(x)$ denote its periodic extension of period 1. It is known that $\bar{B}_m(0) = B_m$ for all $m \geqslant 2$ and that $\bar{B}_m(\frac{1}{2}) = 0$ if $m$ is odd [2, Sections 2 and 3]. These remarks establish the following theorem.

THEOREM 8. (1) *The monospline*

$$\bar{B}_{2k}(x) - B_{2k} \tag{1.17}$$

*is uniquely characterized among all monosplines of degree* $2k$ *by the two requirements* (1.14) *and* (1.16).

(2) *The monospline*

$$\bar{B}_{2k+1}(x + \tfrac{1}{2}) \tag{1.18}$$

*is uniquely characterized among all monosplines of degree* $2k + 1$ *by the two requirements* (1.14) *and* (1.16).

*Finally, the following identities hold*

$$x^m = \sum_{-\infty}^{\infty} \nu^m L_m(x - \nu) + \begin{cases} \bar{B}_m(x) - B_m & \text{if } m \text{ is even}, \\ \bar{B}_m(x + \frac{1}{2}) & \text{if } m \text{ is odd}. \end{cases} \tag{1.19}$$

Alternatively, the Bernoullian functions can be *defined* as solutions of this problem, i.e., by cardinal spline interpolation of $x^m$, and all their properties developed starting from this definition.

An entirely different characterization of $\bar{B}_m(x)$ among monosplines was recently given in [5].

## 2. Proof of the Unicity in Theorem 1

Theorem 1 being obvious if $m = 1$, we shall assume that $m \geqslant 2$.

Let

$$S_m^0 = \{S(x);\ S(x) \in S_m,\ S(\nu) = 0 \text{ for all integers } \nu\}. \tag{2.1}$$

In [4, Section 9] we proved a Theorem 11 which asserts the following:

*If* $1 \leqslant p \leqslant \infty$ *and if*

$$S(x) \in S_m^0 \cap L_p, \tag{2.2}$$

then

$$S(x) = 0 \text{ for all } x. \tag{2.3}$$

The method used in [4] to establish this proposition will easily allow us to prove the following

Lemma 2. *If*

$$S(x) \in S_m^0 \cap F^*, \tag{2.4}$$

*then again* (2.3) *holds.*

We see that the assumption (2.2) has been replaced by the new assumption (2.4).

*Proof.* In [4, relation (9.14)] it is shown that every element of $S_m^0$ admits a unique representation

$$S(x) = \sum_{\nu=1}^{2s} a_\nu S_\nu(x), \qquad \text{where} \quad 2s = \begin{cases} m-2 & \text{if } m \text{ is even,} \\ m-1 & \text{if } m \text{ is odd,} \end{cases} \tag{2.5}$$

where the $S_\nu(x)$ are the so-called *eigensplines* of the class $S_m^0$. The properties of these eigensplines; in particular, the relations (9.18) of [4], and the magnitudes of the corresponding eigenvalues $\lambda_\nu$, show easily that if the function (2.5) is in $F^*$, hence in some $F_s$, then all the coefficients $a_\nu$ must vanish, and therefore (2.3) holds.

Returning to the unicity in Theorem 1, let $S_1(x)$ and $S_2(x)$ be two solutions of the problem (13). It follows that

$$S_1(x) - S_2(x) \in S_m{}^0 \qquad \text{and} \quad S_1(x) - S_2(x) \in F^*$$

and therefore

$$S_1(x) = S_2(x) \text{ for all } x,$$

by Lemma 2.

## 3. The Fundamental Function $L_m(x)$

Since Theorem 1, beyond unicity, is not yet available at this point, the function $L_m(x)$ will now be constructed directly, as already done [3]. It was shown in [4, formula (2.2) and Lemma 6] that the real cosine polynomial

$$\phi_m(u) = \sum_{|\nu| \leqslant m/2} M_m(\nu)\, e^{i\nu u} \tag{3.1}$$

is positive for all real $u$. Also that its reciprocal has a Fourier series expansion

$$\frac{1}{\phi_m(u)} = \sum_\nu \omega_\nu^{(m)} e^{i\nu u}, \tag{3.2}$$

with the following property: The series $\sum_\nu \omega_\nu^{(m)} z^\nu$ is the Laurent expansion on $|z| = 1$ of a rational function having no poles on the circle $|z| = 1$ [4, p. 182]. This implies the existence of an inequality of the form

$$|\omega_\nu^{(m)}| \leqslant Ce^{-\gamma|\nu|}, \qquad \text{for all } \nu, \tag{3.3}$$

for appropriate positive $C$ and $\gamma$ depending on $m$.

We now define the spline function

$$L_m(x) = \sum_\nu \omega_\nu^{(m)} M_m(x - \nu). \tag{3.4}$$

From the mutually reciprocal Fourier expansions (3.1) and (3.2), we obtain by multiplication the relations

$$\sum_\nu \omega_\nu^{(m)} M_m(j - \nu) = \delta_j = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j \neq 0. \end{cases} \tag{3.5}$$

In terms of (3.4), (3.5) may be written as

$$L_m(j) = \delta_j, \tag{3.6}$$

and we conclude that the function (3.4) has the desired property (16).

## 4. Proof of Theorem 2

Using (3.3), the inequalities $0 \leqslant M_n(x) \leqslant M_m(0)$, and that $M_m(x) = 0$ if $|x| \geqslant m/2$, we find from (3.4) that

$$\begin{aligned} |L_m(x)| &\leqslant \sum_{\nu > x - m/2} |\omega_\nu^{(m)}| M_m(x-\nu) \\ &\leqslant M(0) \sum_{\nu > x - m/2} |\omega_\nu^{(m)}| < M_m(0)\, C \sum_{\nu > x - m/2} e^{-\gamma\nu}. \end{aligned}$$

If $x \geqslant m/2$ we easily conclude that

$$|L_m(x)| < c_1 e^{-\gamma x},$$

where $c_1 = M_m(0)\, C(1 - e^{-\gamma})^{-1} e^{m\gamma/2}$. The function $L_m(x)$ being even, this implies (17) with $\gamma_m = \gamma$. That $L_m(x)$ is uniquely defined by the properties (16) and (17) we already know from the already established unicity in Theorem 1.

## 5. Proof of Theorem 3 and the Proof of Theorem 1 Completed

Both these proofs are simultaneously carried out by showing that

$$S(x) = \sum_\nu y_\nu L_m(x - \nu) \tag{5.1}$$

converges locally uniformly and furnishes a solution of the problem (13). By our assumption (14) we may write $|y_\nu| < A(|\nu|^s + 1)$ for all $\nu$, and Theorem 2 shows that the series (5.1) is termwise dominated by the series

$$\sum_\nu AC_m(|\nu|^s + 1)\, e^{-\gamma|x-\nu|}, \tag{5.2}$$

which evidently converges uniformly in $x$ in every finite interval $|x| \leqslant K$. Therefore (5.1) converges locally uniformly and defines an element of $S_m$ satisfying the relation $S(\nu) = y_\nu$, for all $\nu$, in view of (16).

There remains to show that $S(x) \in F_s$ or

$$\sum_\nu y_\nu L_m(x - \nu) = O(|x|^s) \quad \text{as} \quad x \to \pm\infty. \tag{5.3}$$

By (5.2) we see that it suffices to show that

$$\sum_\nu |\nu|^s e^{-\gamma|x-\nu|} = O(|x|^s) \quad \text{as} \quad x \to \pm\infty \tag{5.4}$$

and by symmetry we may establish this statement only if $x \to +\infty$. As the sum of the series (5.4) for negative values of $\nu$ is clearly $= 0(1)$, it is enough to show that

$$\sum_{\nu=1}^{\infty} \nu^s e^{-\gamma|x-\nu|} = O(x^s) \quad \text{as} \quad x \to +\infty. \tag{5.5}$$

We have

$$\begin{aligned} \sum_{\nu=1}^{\infty} \left(\frac{\nu}{x}\right)^s e^{-\gamma|x-\nu|} &= \sum_{\nu \leqslant x+1} + \sum_{\nu > x+1} \\ &< \sum_{\nu \leqslant x+1} e^{-\gamma(x-\nu)} + \sum_{\nu > x+1} \nu^s x^{-s} e^{-\gamma|x-\nu|} \\ &= O(1) + x^{-s} \sum_{\nu > x+1} \nu^s e^{-\gamma(\nu - x)} \\ &= O(1) + x^{-s} e^{\gamma x} \sum_{\nu > x+1} \nu^s e^{-\gamma\nu}. \end{aligned} \tag{5.6}$$

If we restrict $x$ to a range $x > \xi$ in which the function $x^s e^{-\gamma x}$ is decreasing and convex, then we may replace the last sum by an integral and obtain

$$\begin{aligned} x^{-s} e^{\gamma x} \sum_{\nu > x+1} \nu^s e^{-\gamma\nu} &< x^{-s} e^{\gamma x} \int_x^\infty t^s e^{-\gamma t}\, dt \\ &= \int_x^\infty (t/x)^s e^{-\gamma(t-x)}\, dt = \int_0^\infty \left(1 + \frac{u}{x}\right)^s e^{-\gamma u}\, du \end{aligned}$$

by the change of variables $t = x + u$. The last integral being $O(1)$ as $x \to +\infty$, we see by (5.6) that (5.5) holds.

## 6. Proof of Theorem 5

We need a result from [3] concerning the so-called method of *central interpolation* which is as follows. Let $(y_\nu)$ be a sequence of data. We select

$m$ consecutive ordinates, say $y_r, y_{r+1}, \ldots, y_{r+m-1}$, and construct the Lagrange polynomial $P_r(x)$, of degree $m - 1$, that interpolates them and define the interpolating function $F(x)$ by the relation

$$F(x) = P_r(x)$$

within the interval $(\alpha_r, \beta_r)$, of length unity, whose midpoint coincides with the midpoint of the interval $(r, r + m - 1)$. Varying $r$ over all integers we see that $F(x)$ is defined by a piecewise polynomial function. Applying this method to the "unit" sequence $y_\nu = \delta_\nu$, defined by (15), we obtain an interpolating function that we denote by $C_m(x)$. It is readily seen [3, p. 57, where the graphs of $C_m(x)$, for $m = 1, 2, 3, 4$, are also to be found] that the interpolant for an arbitrary sequence $(y_\nu)$ is given by the formula

$$F(x) = \sum_{-\infty}^{\infty} y_\nu C_m(x - \nu).$$

Evidently, from the definition of $C_m(x)$ as the interpolant of $(\delta_\nu)$, we find that

$$C_m(\nu) = \delta_\nu \text{ for all } \nu. \tag{6.1}$$

We shall need

LEMMA 3. *In terms of the B-splines $M_m(x)$ and the coefficients of the expansion* (27) *we have the identity*

$$C_m(x) = \sum_{r=0}^{s} (-1)^r \gamma_{2r}^{(m)} M_m^{(2r)}(x), \qquad \textit{where} \quad s = \left[\frac{m-1}{2}\right]. \tag{6.2}$$

For proof, see [3, Part B, Section 2.3].

*Remark.* Observe that if $m$ is odd then functions with discontinuities of the first kind appear on both sides of (6.2). The identity (6.2) holds for all real $x$ if we normalize discontinuous functions like $C_{2k+1}(x)$ and $M_m^{(m-1)}(x)$ by requiring that

$$f(x) = \tfrac{1}{2}(f(x + 0) + f(x - 0)) \text{ for all } x.$$

In order to establish the relation (28) we start from (24) or

$$S(x) = \sum_\nu c_\nu M_m(x - \nu) \tag{6.3}$$

and obtain by differentiations for integer $x = n$

$$S^{(2r)}(n) = \sum_\nu c_\nu M_m^{(2r)}(n - \nu) \qquad (r = 0, 1, \ldots, s), \tag{6.4}$$

where $s = [(m-1)/2]$. Multiplying both sides by $(-1)^r \gamma_{2r}^{(m)}$ and summing over $r$ we get

$$\sum_{r=0}^{s} (-1)^r \gamma_{2r}^{(m)} S^{(2r)}(n) = \sum_{r=0}^{s} (-1)^r \gamma_{2r}^{(m)} \sum_{\nu} c_\nu M_m^{(2r)}(n-\nu)$$

$$= \sum_{\nu} c_\nu \sum_{r=0}^{s} (-1)^r \gamma_{2r}^{(m)} M_m^{(2r)}(n-\nu)$$

$$= \sum_{\nu} c_\nu C_m(n-\nu) = c_n$$

in view of the relations (6.2) and (6.1). This completes a proof of Theorem 5.

## 7. Proof of Theorem 4

For the simple proof of the sufficiency of the condition (26) see [4, pp. 199–200]. Let us now assume that

$$S(x) \in L_p \tag{7.1}$$

and show that the $c_\nu$ in (6.3) satisfy

$$(c_\nu) \in l_p . \tag{7.2}$$

Let $R(x)$ be a polynomial of degree $k$ in the interval [0, 1]. By Markov's theorem we obtain the string of inequalities

$$\max | R' | \leqslant 2k^2 \max | R | ,$$

$$\max | R'' | \leqslant 2(k-1)^2 \max | R' | ,$$

$$\vdots$$

$$\max | R^{(r)} | \leqslant 2(k-r+1)^2 \max | R^{(r-1)} | , \qquad (r \leqslant k),$$

and multiplying them together we obtain

$$\max | R^{(r)} | \leqslant A(k, r) \max | R | \qquad (r = 1,..., k) \tag{7.3}$$

where $A(k, r) = 2^r(k(k-1) \cdots (k-r+1))^2$.

Let $p = \infty$ in (7.1), which means that $S(x)$ is bounded. Applying (7.3), with $k = m-1$, to each of the polynomial components of $S(x)$ in each of the successive interval $[\nu, \nu+1]$, we conclude that the $m$ sequences

$$(S^{(r)}(\nu)), \ (r = 0, 1,..., m-1)$$

are bounded. Since in (28) $2s \leqslant m - 1$ we conclude from (28) that the sequence $(c_\nu)$ is also bounded so that (7.2) holds.

Let now $1 \leqslant p < \infty$. Assuming $P(x) \in \pi_{m-1}$ and setting

$$R(x) = \int_0^x P(t)\, dt \qquad \text{in} \quad 0 \leqslant x \leqslant 1,$$

we apply (7.3) to this polynomial of degree $k = m$ and obtain for $r \leqslant m - 1$

$$\max | P^{(r)} | = \max | R^{(r+1)} | \leqslant A(m, r + 1) \cdot \max | R |$$

$$= A \cdot \max \left| \int_0^x P(t)\, dt \right| \leqslant A \cdot \int_0^1 | P(t)|\, dt.$$

Hence

$$\max | P^{(r)} | \leqslant A \cdot \left( \int_0^1 | P(t)|^p\, dt \right)^{1/p}$$

by Hölder's inequality. Therefore,

$$\max | P^{(r)} |^p \leqslant A^p \cdot \int_0^1 | P(t)|^p\, dt \qquad (r = 0,..., m - 1). \tag{7.4}$$

Assume for the moment that $m$ is even and observe that in (28) $2s = m - 2$. Applying (7.4) to the components of $S(x)$ in successive intervals $[\nu, \nu + 1]$ and summing the results, we obtain

$$(\| S^{(r)}(\nu)\|_p)^p \leqslant \sum_\nu \max_{[\nu,\nu+1]} | S^{(r)}(x)|^p \leqslant A^p \cdot \int_{-\infty}^{\infty} | S(t)|^p\, dt < \infty$$

$$(r = 0, 1,..., m - 2)$$

and this shows that the $s + 1$ sequences appearing on the right side of (28) are all in $l_p$. It follows that $(c_\nu) \in l_p$.

If $m$ is odd, then in (28) $2s = m - 1$. Applying (7.4) to the components of $S(x)$ in the intervals $[\nu - \frac{1}{2}, \nu + \frac{1}{2}]$ we obtain similarly that

$$(\| S^{(r)}(\nu)\|_p)^p \leqslant \sum_\nu \max_{[\nu-\frac{1}{2},\nu+\frac{1}{2}]} | S^{(r)}(x)|^p < A^p \int_{-\infty}^{\infty} | S(t)|^p\, dt < \infty$$

$$(r = 0, 1,..., m - 1)$$

and (26) is thereby similarly established.

*Added in proof:* Theorem 1 for the case when $m$ is even and $s = 0$ was first established by Ju. N. Subbotin in 1965 in the paper quoted among the references of [4] above.

## References

1. A. Cavaretta, On cardinal perfect splines of least sup norm on the real axis, Ph.D. thesis, University of Wisconsin, 1970.
2. N. E. Nörlund, "Differenzenrechnung," Springer Verlag, Berlin, 1924.
3. I. J. Schoenberg, Contributions to the problem of approximation of equidistant data by analytic functions, *Quart. Appl. Math.* **4** (1946), 45–99, 112–141.
4. I. J. Schoenberg, Cardinal interpolation and spline functions, *J. Approximation Theory* **2** (1969), 167–206.
5. I. J. Schoenberg and Z. Ziegler, On cardinal monosplines of least $L_\infty$-norm on the real axis, *J. Anal. Math.* **23** (1970), 409–436.
6. I. J. Schoenberg and A. Cavaretta, Solution of Landau's problem concerning higher derivatives on the halfline, MRC Tech. Sum. Rep. #1050, University of Wisconsin, March 1970; also, Proceedings of the Conference on Constructive Function Theory, Varna, Bulgaria, May 1970.